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Classical and Quantum Probability in the ϵ-Model

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We describe the probabilistic study of a hidden variable model in which the origin of the quantum probability is due to fluctuations of the internal state of the measuring apparatus. By varying the intensity of these fluctuations from zero to a maximal value, we describe in a heuristic manner the transition from classical behavior to quantum behavior. We characterize this transition in terms of the Accardi–Fedullo inequalities. This is a review article in which we gather our recent contributions to the subject, most of which have not been published in article form.

1. INTRODUCTION

In this article we present a probabilistic study of a model with two possible outcomes related to each measurement which allows a quantum mechanical as well as a classical description (see Section 2). Whenever we use the words classical or quantum, we mean that the *probabilities* related to a measurement are the same as those that can be computed by these respective theories. For example, the quantum probability of our model is the same as the transition probability of a Stern–Gerlach measurement on a spin-1/2 particle. For the classical case we have to distinguish between the deterministic case and the Kolmogorovian case. We will say deterministic whenever the probabilities of the model are either 0 or 1, and Kolmogorovian when the probabilities are regarded as a measure on a σ-algebra of subsets of the sample space (this point will be made more precise later).

Our model is able to reproduce the quantum mechanical transition probabilities by assuming that there is a lack of knowledge about the interaction between the system that we study and the measurement apparatus. By intro-

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ducing a parameter ϵ, we can change the magnitude of this lack of knowledge. If this lack of knowledge about the interaction is absent, we recover the deterministic case. To recover the Kolmogorovian probabilities, we introduce Aerts' definition of a conditional probability. If we look at the Aerts conditional probabilities that arise when there is no lack of knowledge about the interaction (the deterministic case), we recover the Kolmogorovian case. We have studied the intermediate behavior of this model within various frameworks such as the lattice-theoretic approach (D. Aerts and Durt, 1994a; Durt, 1996a; D. Aerts *et al.*, 1997), the algebraic approach (D. Aerts and D'Hooghe, 1996) and the probabilistic approach (S. Aerts, 1994, 1996, 1998; Lévêque, 1995; Durt, 1996a). In this approach, we examine the transition and conditional probabilities associated with our model with the help of the Accardi–Fedullo inequalities. These inequalities (see Section 3) express the existence of a Kolmogorovian model or of a Hilbert space model for a triple of probabilities. If we apply these inequalities to the transition probability deduced from the ϵ model, the result is simple (see Section 4), but rather surprising (Durt, 1996a): no Kolmogorovian model exists for the transition probability, neither in the deterministic situation nor in the quantum nor any intermediate situation; no Hilbert space model exists except for the quantum case. In the case of the Aerts conditional probability defined in Section 5, we show that we do have perfect agreement between the Accardi–Fedullo classification and ours: the conditional probability admits a Kolmogorovian model in the classical limit and a Hilbert space model in the quantum limit. These cases are also limiting cases for the Accardi–Fedullo inequalities because the conditional probabilities associated with the respective limiting cases saturate the corresponding inequalities and violate them in neighboring intermediate cases (Durt, 1996a).

In Section 6, we generalize the model to what we call the η-ϵ-model, and discuss the classical limit in this context. In particular we show, following Lévêque (1995), that we must replace the Kolmogorovian model defined by Accardi and Fedullo by a generalized Kolmogorovian model, and that the existence of such a model is expressed by three inequalities instead of four as in the Accardi–Fedullo theorem. We show that the η-ϵ model fulfills this reduced set of inequalities.

2. THE ϵ MODEL

As stated in the introduction, the ϵ-model covers a very broad spectrum of probabilities for experiments with only two possible outcomes, i.e., a yes–no experiment or a spin measurement of a spin-1/2 particle (e.g., the electron). Let us take the example of the spin measurement as our generic quantum experiment and neglect all other properties that the electron might have. We know that the state of a spin-1/2 particle can be expressed as a superposition of a spin-up and a spin-down state along an *a priori* direction z: $|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$. In the ϵ-model, the states of the system that we wish to study are represented as points on the unit sphere. That the set of points on the unit sphere covers all possible spin-1/2 states of the two-dimensional Hilbert space is demonstrated by the Pauli mapping, which maps bijectively the set of physical states onto the unit 3-sphere: $|\psi\rangle \rightarrow n = (2 \text{ Re } \alpha^*\beta, 2 \text{ Im } \alpha^*\beta, |\alpha|^2 - |\beta|^2)$ (in Cartesian coordinates) and conversely $\alpha = \cos(\theta/2)e^{-i\varphi/2}$, $\beta = \sin(\theta/2)e^{i\varphi/2}$, where θ, φ are the polar angles of n.

2.1. Guiding Principles of the ϵ-Model

We want to introduce the ϵ-model independently of the physical theories of quantum mechanics and classical mechanics, because we want to make statements about these theories and hence need this independence. At the same time we do not want to introduce the ϵ-model in a purely formal way, since then it could be thought that it can be realized just as a formal structure. To show that the probability structure that we find in the ϵ-model (which will contain quantum probability and classical probability as special cases, but also generates a structure that is neither quantum not classical) can correspond to 'real' probabilities, appearing in our reality as a limit of the relative frequency of repeated experiments on systems prepared in an identical way, we introduce the ϵ-model by means of a simple mechanical model of which the functioning—as will be explained in this section—generates this probability structure.

In order to introduce our ϵ-model we need three basic concepts: states to characterize the entity we wish to study, measurements that can be applied to this state in order to gain information about the state, and a rule that tells us how the state transforms upon measurement and how to assign the outcome of a measurement when it is applied to a certain state. The set of states Σ that characterizes the property we wish to measure consists of the points of the unit sphere. To represent a state we shall write p_v, where v denotes the unit vector that represents the state of the entity at the moment of measurement: $\Sigma = \{p_v | v \text{ is on the unit sphere}\}$. For each point u on the sphere, we introduce the following experiment eU. We consider the diametrically opposite point $-u$, and install an elastic band of length 2 such that it is fixed with one of its endpoints at u and the other endpoint at $-u$ (see Fig. 1). The elastic band will be called "ϵ-elastic" because it consists of two different parts: an unbreakable part $[-1, -\epsilon[\text{U}] + \epsilon, +1]$ and a breakable part $[-\epsilon, +\epsilon]$, with $\epsilon \in [0, 1]$. Once the ϵ-elastic is installed, the state p_v is projected from its original place v orthogonally onto the wire and sticks on it. Then the ϵ-elastic breaks and the state attached to either one of the two parts is

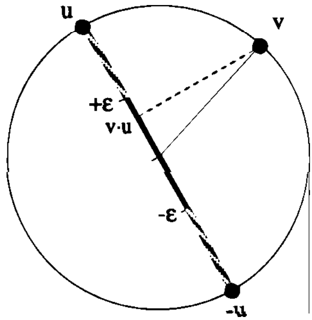

Fig. 1. A representation of the ϵ-model and the experiment. The elastic breaks uniformly inside the interval $[-\epsilon, +\epsilon]$, and is unbreakable outside this interval, at the points of the set $[-1, -\epsilon[\cup] +\epsilon, 1]$.

'dragged' to one of the two endpoints u or $-u$. Depending on whether the state p_v ends up at u or at $-u$, we give the outcome o_1^u or o_2^u to e_u and the state transforms correspondingly to either p_{+u} or p_{-u}. We can easily calculate the probabilities corresponding to the two possible outcomes. The state p_v is transformed into the state p_u when the elastic part breaks at a point of the interval $L_1 = [-\epsilon, u \cdot v]$ (which is the length of the piece of the elastic between $-u$ and the point onto which the state was projected), and arrives at $-u$ when it breaks at a point of the interval $L_2 =]u \cdot v, +\epsilon]$. We make the hypothesis that the elastic breaks in a uniform way, which means that the probability that the state p_v is transformed to the state p_u is given by the length of L_1 (which is $\epsilon + \cos\theta$) divided by the total length of the elastic (which is 2ϵ). The probability that the state p_v is transformed into the state p_{-u} is the length of L_2 (which is $\epsilon - \cos\theta$) divided by the total length of the elastic. To summarize, we have:

(1) $v \cdot u \leq -\epsilon$. The state p_v is projected onto the lower part of the ϵ-elastic, and *any* breaking of the elastic will transform it into the state p_{-u}. We have $P(o_1^u, p_v) = 0$ and $P(o_2^u, p_v) = 1$.

(2) $-\epsilon < v \cdot u < \epsilon$. The state p_v is projected onto the breakable part of the ϵ-elastic. We can easily calculate the transition probabilities and find

$$P(o_1^u, p_v) = \frac{1}{2\epsilon}(v \cdot u + \epsilon)$$

$$P(o_2^u, p_v) = \frac{1}{2\epsilon}(\epsilon - v \cdot u)$$

(3) $\epsilon \leq v \cdot u$. The state p_v is projected onto the upper, unbreakable part of the ϵ-elastic, and any breaking of the elastic will pull it upward such that it arrives at u. We have $P(o_1^u, p_v) = 1$ and $P(o_2^u, p_v) = 0$.

3. STATISTICAL INEQUALITIES AS CONDITIONS OF REPRESENTABILITY

More than a century ago, George Boole (Boole, 1862), the founder of modern logic, wrote:

> Let $p_1, p_2, \ldots, p_n$ represent the probabilities given in the data. As these will in general not be the probabilities of unconnected events, they will be subject to other conditions than that of being positive. . . Those other conditions will, as will hereafter be shown, be capable of expressions by equations or in equations reducible to the general form: $a_1 \,.\, p_1 + a_2 \,.\, p_2 + \ldots + a_n \,.\, p_n + a \geq 0$, $a_1, \ldots, a_n, a$ being numerical constants which differ for the different conditions in question. These, together with the former, may be termed conditions of possible experience. When satisfied they indicate that the data may have, when not satisfied they indicate that the data cannot have resulted from an actual observation.

More than half a century later the mathematician Bonferroni (Gallambos and Simonelli, 1996) would construct specific instances of such inequalities that are now known as Bonferroni inequalities. The inequalities arise because of the mathematical structure of classical probability, where probabilities are regarded as measures on a σ-algebra of subsets of a set called the event space. It is ultimately the logic of sets which dictates the constraints of classical (or what we now call Kolmogorovian) probability theory. The first to realize that, because of the vector space structure of Hilbert space and the specific form of the transition probability as the squared modulus of an inner product, the quantum mechanical probabilities are also bound by inequalities was Bogdan Mielnik (Mielnik, 1968). So we have the following situation: probabilities from quantum mechanics and Kolmogorovian probability theory both have to comply with constraints. In both situations the constraint is expressible as an inequality or a set of inequalities, but the specific form of the inequalities depends on the axioms that lead to the inequalities (classical or quantum). That is why we prefer to label the inequalities as "conditions of representability" rather than "conditions of possible existence." In 1982, Accardi and Fedullo generalized Mielnik's work by deducing inequalities which express whether a set of data can be represented by a real or a complex Hilbert space. The fact that one can distinguish not only between a classical and a quantum probabilistic model underlying the data, but even between a real and a complex Hilbert space shows the resolvent power of this technique. Accardi and Fedullo also derived "classical" inequalities to reveal the existence of a Kolmogorovian model for conditional probabilities and showed that the quantum probabilities do not admit such a model. The "classical" inequalities of Accardi and Fedullo are in fact a direct generalization of the Gutkowski–Masotto inequalities (Gutkowski and Masotto, 1972), which themselves can be shown to be equivalent to Bell's inequality (Corleo *et al.*, 1975). In 1989, Pitowski showed the equivalence between generalized "classical" inequalities and Clauser–Horne inequalities, a variant of Bell's inequalities particularly well adapted to the experimental situation in which violations occur. Because we will make extensive use of the Accardi–Fedullo inequalities, we shall now present a summary of their work. Then we shall apply these results to the ϵ-model.

3.1. The Kolmogorovian Model for Three Conditional Probabilities

Let us consider three dichotomic experiments A, B, C, with the outcomes A_+, A_-, B_+, B_-, C_-, C_+. Let us introduce the shorthand notation $P(X_+|Y_+)$ for the conditional probability $P(X = X_+|Y = Y_+)$ of getting the result X_+ when measuring the observable X, when the obervable Y is known to take the outcome Y_+. We can construct 36 conditional probabilities among these six possible outcomes, of which 24 are *a priori* unknown because the requirement of dichotomy implies that $P(X_+|X_-) = P(X_-|X_+) = 0 = 1 - P(X_+|X_+) = 1 - P(X_-|X_-)$.

The conditional probabilities are said to admit a Kolmogorovian model iff:

(i) There exists a probability space that is characterized by a triple (Ω, Σ, μ) where Ω is a nonempty set, Σ the σ-algebra of subsets of Ω, and μ a probability measure on Σ.

(ii) For each observable, there exists a measurable partition of Ω (for instance, for A, we have A_+, A_-: $A_+ \cap A_- = \emptyset$, $A_+ \cup A_- = \Omega$).

(iii) The conditional probability is given by the Bayes formula: $P(A_+|B_+) = \mu(A_+ \cap B_+)/\mu(B_+)$.

The criterion for the existence of a Kolmogorovian model is the content of a theorem of Accardi and Fedullo (1982):

Theorem 1. If the conditional probability is symmetrical $[P(A_+|B_+) = P(B_+|A_+), P(A_-|B_+) = P(B_+|A_-) \ldots]$, it admits a Kolmogorovian model iff the three conditional probabilities p, q, r [respectively $P(A|B)$, $P(B|C)$, $P(C|A)$] fulfill the inequalities

$$|p + q - 1| \leq r \leq 1 - |p - q| \qquad (1)$$

Comments. (a) If the probability is symmetrical, the three conditional probabilities p, q, r define the whole set of probabilities. For instance, $P(B_+|A_-) = P(A_-|B_+) = 1 - P(A_+|B_+) = 1 - P(B_+|A_+) = P(B_-|A_+) = P(A_+|B_-) = 1 - p$ and $P(B_+|A_+) = P(A_+|B_+) = 1 - P(A_+|B_-) = 1 - (1 - P(A_-|B_-)) = P(A_-|B_-) = P(B_-|A_-) = p$.

(b) Apparently, the inequalities of Accardi–Fedullo seem to privilege one probability (r), but this is only a formal appearance: the inequalities are in fact invariant under any permutation of the triplet (p, q, r).

3.2. The Hilbert Space Model

We shall again consider the triplet of conditional probabilities (p, q, r), which allows us to define completely the 24 conditional probabilities when they are symmetrical, as we saw before. These probabilities are said to admit a Hilbert space representation iff there exist three normalized vectors (states) A_+, B_+, C_+ of a two-dimensional Hilbert space such that $p = |\langle A_+|B_+\rangle|^2$, $q = |\langle B_+|C_+\rangle|^2$. $r = |\langle A_+|C_+\rangle|^2$, where $|\langle A_+|B_+\rangle|^2$ is the squared modulus of the Hilbert scalar product between the states A_+ and B_+. Note that the requirement of symmetry is automatically fulfilled, because of the symmetry of the inner product in Hilbert space. With A_-, B_-, C_- we can associate vectors of the Hilbert space orthogonal to A_+, B_+, C_+ so that we recover the complementary probabilities. We are in accordance with the axioms of quantum mechanics, which tell us that two eigenstates corresponding to different results of the same observable are necessarily orthogonal.

The possibility of existence of a Hilbert space model is the object of a second theorem (Mielnik, 1968; Accardi and Fedullo, 1982):

Theorem 2. If the conditional probability is symmetrical, it admits a Hilbert space model iff the three conditional probabilities p, q, r [respectively $P(A_+|B_+)$, $P(B_+|C_+)$, $P(C_+|A_+)$] fulfill the inequalities

$$(\sqrt{pq} - \sqrt{(1-p)(1-q)})^2 \leq r \leq (\sqrt{pq} + \sqrt{(1-p)(1-q)})^2 \quad (2)$$

An important corollary is that the inequalities for the existence of a Kolmogorovian model are stronger than the inequalities related to a Hilbert space model, in the sense that whenever the first are fulfilled, the second are fulfilled, too. This is a consequence of the following inequalities:

$$\begin{aligned} (\sqrt{pq} - \sqrt{(1-p)\,(1-q)})^2 &\leq |p + q - 1| \\ 1 - |p - q| &\leq (\sqrt{pq} + \sqrt{(1-p)\,(1-q)})^2 \end{aligned} \quad (3)$$

We thus have the following situation: The three conditional probabilities (p, q, r) define a point in the unit cube $[0, 1] \times [0, 1] \times [0, 1]$. Let us call the classical zone the set of triplets (p, q, r) satisfying the classical inequalities (as defined in theorem 1); then, all the triplets of the classical zone admit a Kolmogorovian model. Let us call the quantum zone the set of triplets (p, q, r) satisfying the quantum inequalities (as defined in Theorem 2); then, all the triplets of the quantum zone admit a Hilbert space model. The classical zone is included in the quantum zone. Note that the quantum zone does not cover the cube. The triplet (0.99, 0.99, 0.9) for instance, violates the quantum inequalities.

4. THE ACCARDI–FEDULLO INEQUALITIES AND THE ϵ-MODEL

We shall analyze in this section the transition probability related to the ϵ-model in terms of the Accardi–Fedullo inequalities and discuss whether it admits a Kolmogorovian model or a Hilbert space model.

As we saw in the previous section, the probability of occurrence of the result "spin up" for a measurement when θ is the angle between the direction labeled A_+ of the measuring apparatus and the state as represented on the Poincaré sphere B_+ is 1 when $\cos\theta > \epsilon$, 0 when $\cos\theta < -\epsilon$, and intermediate in between. Note that when $\epsilon = 0$, then $P(\theta) = \cos^2(\theta/2)$, the quantum probability. The ϵ-transition probability is symmetrical in the angle between A_+ and B_+, that is, $P(A_+|B_+) = P(B_+|A_+)$. The following theorem characterizes the probability $P(A_+|B_+)$ in terms of the Accardi–Fedullo inequalities (Durt, 1996a):

Theorem 3. The $\in$ transition probability does not admit a Hilbert space model unless $\epsilon = 1$ (the quantum case). It never admits a Kolmogorovian model.

Proof. If $\epsilon \neq 1$, then it is always possible to find three states (we identify a state with a point of the sphere) A, B, C such that $P(A_+|B_+) = P(B_+|C_+) = 1$, $P(A_+|C_+) \neq 1$. Identify A_+, B_+, and C_+ with three points of the same great circle on the sphere, the angle between A_+ and B_+ smaller than but nearly equal to $\arccos\epsilon$ [so $P(A_+|B_+) = 1$], and the angle between C_+ and A_+ larger than, but nearly equal to $\arccos\epsilon$ [so $P(A_+|C_+) \neq 1$], so that the angle between B_+ and C_+ is smaller than $\arccos\epsilon$ [so $P(B_+|C_+) = 1$]. If we identify p, q, r with $P(A_+|B_+)$, $P(B_+|C_+)$, $P(A_+|C_+)$, and replace p and q by 1 in the inequality $(\sqrt{pq} - \sqrt{(1-p)(1-q)})^2 \leq r \leq (\sqrt{pq} + \sqrt{(1-p)(1-q)})^2$, we obtain $1 \leq r \leq 1$, so $r = 1$. But here $r = P(A_+|C_+) \neq 1$, so that we violate the inequalities of Accardi–Fedullo. Thus, the ϵ-probability admits no Hilbert space model (and *a fortiori* no Kolmogorovian model) in this case. It is obvious that in the quantum case the probability admits a Hilbert space model. This is straightforward after identification of the points of the sphere and their image under the Pauli mapping. As a corollary, the ϵ-probability does not admit a Kolmogorovian model unless perhaps if $\epsilon = 1$ (the quantum case). But even then, as was already noticed by Accardi and Fedullo, it does not admit such a model. To show this, take the following choice for A, B, C: identify A_+, B_+, C_+ with three points belonging to the same great circle, the angle between A_+ and B_+ equal to $\pi/3$, between B_+ and C_+ also equal to $\pi/3$, but between C_+ and A_+ equal to $2\pi/3$. Then $p = 3/4$, $q = 3/4$, $r = 1/4$, and the first inequalities, necessary for the existence of a Kolmogorovian model, are violated. For instance, $|p + q - 1| = 1/2 \nless 1/4 = r$, in contradiction with the required inequality.

The proof presented here utilizes the inequalities of Accardi and Fedullo in the particular case where two of the conditional probabilities are equal to one. In this case the possibility of existence of a Kolmogorovian model, as well as of a Hilbert space model, expressed by the Accardi–Fedullo inequalities implies that the third conditional probability equals one. We shall give here a direct and intuitive explanation of this condition, without employing the inequalities (D. Aerts and Durt, 1994b). First we assume that the triplet admits a Kolmogorovian representation and denote by A_+ that subset of the event space that corresponds to the outcome A_+ for observable A. We notice that, if $P(A_+|B_+) = 1$, then $\mu(A_+ \cap B_+) = \mu(A_+)$, so that $A_+ \subset B_+$, up to subsets of Ω of null measure. The symmetry of the conditional probability implies that $B_+ \subset A_+$, up to subsets of Ω of null measure by a similar argument. But then $A_+ = B_+$ up to subsets of Ω of null measure. Similarly, $P(B_+|C_+) = 1$ implies that $B_+ = C_+$ up to subsets of Ω of null measure. But then, $A_+ = C_+$ up to subsets of Ω of null measure. In this case, $P(A_+|C_+) = 1$. Next assume that the triplet of probabilities admits a Hilbert space representation. If $P(A_+|B_+) = 1$, then there exist two normalized vectors A_+, B_+ of a two-dimensional Hilbert space of which the modulus of the inner product is one. This implies that $A_+ = B_+$ up to a physically irrelevant phase. Similarly, $P(B_+|C_+) = 1$ implies that $C_+ = B_+$ up to a physically irrelevant phase. Then, $A_+ = C_+$ up to a physically irrelevant phase, and $P(A_+|C_+) = 1$.

5. THE CONDITIONAL PROBABILITY

Accardi introduces, as is usual in classical probability theory, the conditional probability by means of the Bayes axiom. Although the Bayes axiom takes a central place in Kolmogorovian probability theory, it is a nonoperational definition for noncompatible observables because the observables do not take their values simultaneously. Still, it will be clear that the preparation of a state inside a given subset of the set of states can be regarded as a kind of conditioning for any consecutive measurement. Following D. Aerts (1995), we propose a natural extension of the concept of conditional probability that is operational both in the quantum and in the classical regime.

Definition. The conditional probability $P_{cond}(X = x|Y = y)$ is the probability that a measurement of observable X gives the result x when we know that *if* we would choose to measure the observable Y, we would find the result y with certainty.

It is important to note that the transition probability and the conditional probability express two different interpretations of conditioning: in the first case (the transition probability), the condition "B is fulfilled" means that the initial state *is* B_+; in the second case (the conditional probability), the condition "B is fulfilled" means that *if an experiment would be performed* with a measuring apparatus along the direction B, we would find with certainty the result B_+. Although the attentive reader might have anticipated this, we will later show how this definition reduces to the transition probability in the quantum case and to the standard conditional probability in the classical case. In the intermediate cases they are different, so we will write P_{cond} for the conditional probability as defined above, with no subscript for the transition probability.

5.1. The Conditional Probability of the ϵ-Model

Let us denote by $eig\{A\}$ the set of states on the sphere that, upon measurement, all lead to the result A_+ with certainty, and likewise for B. In the case of our model $eig\{B\}$ equals the spherical sector of angular opening $2 \arccos \epsilon$ around B. The definition of conditioning as given above now simply means that the possible set of states upon which is conditioned equals $eig\{B\}$. Therefore we must integrate the ϵ-transition probability $P(A|C)$ with C belonging to $eig\{B\}$ to obtain the conditional probability $P_{cond}(A|B)$. To normalize the result, we divide by $2\pi(1 - \epsilon)$, the surface of this sector. The actual integral can be calculated thanks to a judicious application of Gauss' theorem and of spherical trigonometry. Since this has been published (S. Aerts, 1996) we shall not repeat the calculation. Because we shall make use of it, let us show the final result, which is the expression of the conditional probability as a function of the angle θ between A and B, for each value of ϵ:

$$P_{cond}(\theta, \epsilon) = p_1(\theta, \epsilon) \cdot H\left(\epsilon - \cos\frac{\theta}{2}\right) + H\left(\epsilon - \sin\frac{\theta}{2}\right) \cdot p_2(\theta, \epsilon) \cdot H\left(\cos\frac{\theta}{2} - \epsilon\right) + p_3(\theta, \epsilon) \cdot H\left(\sin\frac{\theta}{2} - \epsilon\right)$$

where $H(x)$ is the Heaviside function and

$$p_1(\theta, \epsilon) = \frac{\cos\theta\,(1+\epsilon)}{4\epsilon} + \frac{1}{2}$$

$$p_2(\theta, \epsilon) = p_1(\theta, \epsilon) + \frac{1}{2} + \frac{\omega(u, w)}{4\pi(1-\epsilon)} + \frac{\cos\theta + 1}{4\pi\epsilon(1-\epsilon)} \cdot \sigma(u, w)$$

$$p_3(\theta, \epsilon) = p_1(\theta, \epsilon) + \frac{\omega(u, w) - \omega(-u, w)}{4\pi(1-\epsilon)} + \frac{(cos\,\theta - 1) \cdot \sigma(-u, w) + (\cos\theta + 1) \cdot \sigma(u, w)}{4\pi\epsilon(1-\epsilon)}$$

where

$$\omega(\theta, \epsilon) = 4\epsilon \arccos \sqrt{\frac{1 - (\epsilon/cos\ (\theta/2))^2}{1 - \epsilon^2}} - 4 \arcsin \left(\frac{\sin(\theta/2)}{\sqrt{(1 - \epsilon^2}} \right)$$

and

$$\sigma(\theta, \epsilon) = \epsilon\ tg \frac{\theta}{2} \sqrt{1 - \left(\frac{\epsilon}{\cos(\theta/2)} \right)^2} - (1 - \epsilon^2) \arccos \left(\frac{\epsilon\ tg(\theta/2)}{\sqrt{1 - \epsilon^2}} \right)$$

It is easy to show (S. Aerts, 1996) that if we set $\epsilon = 1$, we find

$$P_{cond}\ (\theta, \epsilon = 1) = \cos^2(\theta/2)$$

which is the well-known quantum mechanical transition probability: $P_{cond}(\theta, \epsilon = 1) = P(A_+|B_+)$.

If, on the other hand, we take the classical limit (ϵ equal to zero), the conditional probability becomes

$$P_{cond}\ (\theta, \epsilon = 0) = \frac{\pi - \theta}{\pi}$$

Now suppose one is asked for the probability that the system would be found in the upper half of the sphere ($eig\{A\}$) when we know for certain that its state belongs to $eig\{B\}$. If one would apply Bayes' axiom with a uniform probability measure μ, one would come up with the following result:

$$\frac{\mu(eig\{A\} \cap eig\ \{B\})}{\mu(eig\{B\})} = \frac{\pi - \theta}{\pi}$$

which is exactly the result stated above.

5.2. The Conditional Probability and the Accardi—Fedullo Inequalities

To check if the inequalities of Accardi and Fedullo are fulfilled, we must consider all triples of points on the sphere, determine their relative angles, replace the values of these angles in the expression of the conditional probability, and finally implement these values in the inequalities of Accardi and Fedullo. This work was performed by Lévêque (1995). We reproduce here some results that he obtained. He showed numerically that the quantum probability is "isolated": It naturally admits a Hilbert space model, but any conditional probability associated with values of ϵ in the vicinity of 1 (the quantum case) does not admit a Hilbert space model, and as a consequence there does not exist a Kolmogorovian model. He showed also that the classical probability ($\theta_{sup} = 0$) is "isolated": It admits a Kolmogorovian model, but the conditional probability with values of ϵ in the vicinity of 0 (the classical case) does not admit a Kolmogorovian model. Remark that this is not true for the Hilbert space model: there exists a broad zone surrounding the classical zone for which the conditional probability admits a Hilbert space model. To show this is rather technical, and requires the use of a computer, so that it would be tedious to describe it in detail here. Furthermore, for a rigorous proof of these results, a computer is certainly less convincing than purely analytical results, especially for the limiting cases, where infinite precision is required. Durt (1996) proved the following proportions about the classical and the quantum limits:

(i) The conditional probability violates the inequalities related to the existence of a Hilbert space model for all values of ϵ inside an open interval upper bounded by 1 (the quantum case); when $\epsilon = 1$, the inequalities are saturated.

(ii) The conditional probability violates the inequalities related to the existence of a Kolmogorovian model for all values of ϵ inside an open interval lower bounded by 0 (the classical case); when $\epsilon = 0$, the inequalities are saturated.

We reproduce here the proof of the first proposition and give a new proof of proposition (ii). Before making the characterization of the conditional probability obtained here in terms of the Accardi–Fedullo inequalities, it is worth noticing that it is symmetrical: $P_{cond}(A|B) = P_{cond}(B|A)$, essentially because the transition probability $P(A|B)$ depends on the relative angle between A and B only. It will be useful for the proof of (i) to remark that when $\epsilon \geq \cos(\theta/2)$ and $\epsilon \geq \sin(\theta/2)$, the conditional probability is given by (see previous subsection)

$$P_{cond}(\theta) = \frac{(1 + \epsilon) \cos \theta}{4\epsilon} + \frac{1}{2} \tag{4}$$

In the quantum case, ϵ yields 1, so $\epsilon \geq \cos(\theta/2)$ for all values of θ and the conditional probability is equal to the quantum probability, and to the ϵ transition probability as well:

$$P_{cond}(\theta) = \frac{1 + \cos \theta}{2} \tag{5}$$

Before we prove the second statement (ii), we remark that for θ in the neighborhood of zero, $\epsilon - \sin(\theta/2) > 0$ and $\cos(\theta/2) - \epsilon > 0$, so that

$$P_{cond}(\theta) = p_2 = p_1 + \frac{1}{2} + \frac{\omega(u, w)}{4\pi(1 - \epsilon)} + \frac{\cos \theta + 1}{4\pi\, \epsilon(1 - \epsilon)} \sigma(u, w)$$

We shall now show the violation of the inequalities in the neighborhood of the quantum case.

Theorem 4 (Durt, 1996a). For all $\epsilon \in [\cos(\pi/8), 1[$, the conditional probability induced by the symmetrical ϵ-distribution violates the inequalities related to the existence of a Hilbert space model.

Proof. Let us consider three coplanar points A, B, C such that the angle between A and B is θ and the angle between B and C equals β. Then, the angle between A and C equals $\alpha - \beta$ or $\alpha + \beta$. In the quantum case, $p = P(A|B) = P(\alpha) = \cos^2(\alpha/2)$, $q = P(B|C) = \cos^2(\beta/2)$, $r = P(C|A) = \cos^2[(\alpha +/- \beta)/2]$. If we replace these values in the Accardi–Fedullo inequalities $[\sqrt{pq} - \sqrt{(1-p)(1-q)}]^2 \leq r \leq [\sqrt{pq} + \sqrt{(1-p)(1-q)}]^2$ we obtain

$$\begin{aligned} &\|\cos(\alpha/2)\cos(\beta/2)| - |\sin(\alpha/2)\sin(\beta/2)\| \\ &\quad \leq |\cos[(\alpha +/- \beta)/2]| \\ &\quad \leq \|\cos(\alpha/2)\cos(\beta/2)| + |\sin(\alpha/2)\sin(\beta/2)\| \end{aligned}$$

The quantum probability obviously saturates these inequalities.

Beside this, the saturation of the inequalities is in some way maximal when the points A, B, C are coplanar (Lévêque, 1995). In the quantum case, this can be shown as follows: choosing A, B, and C as in the previous example, keeping A and B fixed, we vary C on the circle closing the spherical sector of opening β around B. When C covers the circle, r covers the interval $[(\sqrt{pq} - \sqrt{(1-p)(1-q)})^2 \cdot (\sqrt{pq} + \sqrt{(1-p)(1-q)})^2]$ and reaches its extrema when A, B, and C are coplanar. Such circles cover the sphere. This shows that, in the quantum case, the inequalities are not violated and at best saturated, corresponding to the fact that a Hilbert space model exists in this case. Let us call α the infimum of the domain of the expression $[(1 + \epsilon)\cos\theta]/4\epsilon + 1/2$ introduced before: $\epsilon = \cos(\alpha/2)$. When $\epsilon \in [\cos(\pi/8). 1[$, $0 < \alpha \leq \pi/4$. Let us consider three coplanar points A, B, C such that the angle between A and B is $\pi/4$, while the angle between A and C is $\pi/2$. In this case, the angle between B and C equals $\pi/4$. When ϵ belongs to the interval $[\cos(\pi/8), 1[$, p and q are both equal to

$$\frac{(1+\epsilon)\cos(\pi/4)}{4\epsilon} + \frac{1}{2}$$

while r equals 1/2. If $p = q$, then $[\sqrt{pq} - \sqrt{(1-p)(1-q)}]^2 \leq r \leq [\sqrt{pq} + \sqrt{(1-p)(1-q)}]^2$ becomes $(2p-1)^2 \leq r \leq 1$. As already noticed, the quantum probability saturates this inequality: if $p = q = \cos^2(\pi/8)$, $r = \cos^2(\pi/4)$, then $(2p - 1)^2 = r$. Now, the conditional probability takes the same value r as the quantum probability, but takes a value of A strictly superior to it (and also to 1/2, as can be verified). Because $(1 + \epsilon)/4\epsilon$ is a decreasing function of ϵ in the interval [0, 1], we have that when $\epsilon < 1$, $P_{cond}(\theta) > (1 + \cos\theta)/2$. The function $(2p - 1)^2$ is monotonically increasing when $p \geq 1/2$, so that the conditional probability violates the inequality for this choice of A, B, C. As a consequence, it does not admit a Hilbert space model, and, *a fortiori*, no Kolmogorovian model. Concerning the classical limit, let us prove the following theorem (T. Durt, unpublished proof):

Theorem 5. For ϵ inside the interval $]0, 1/2]$, the conditional probability violates the inequalities related to the existence of a Kolmogorovian model.

Proof. When $0 \neq \epsilon \neq 1$, the conditional probability in the surrounding of the origin is expressed by the function $p_2(\epsilon, \theta)$. A straightforward computation shows that the first derivative of the conditional probability taken in $\theta = 0$ is equal to

$$\frac{\epsilon}{2\pi(\epsilon - 1)} \cdot \sqrt{\frac{1 + \epsilon}{1 - \epsilon}}$$

This function tends to zero when ϵ tends to 0 (the classical limit), and remains larger than $-1/\pi$ when $\epsilon \in \,]0, 1/2]$. Furthermore, since the function p_2 is analytical in θ for each fixed value of ϵ it can be approximated by a first-order Taylor series around the origin:

$$P(\theta) = 1 + \frac{\epsilon}{2\pi(\epsilon - 1)} \cdot \sqrt{\frac{1 + \epsilon}{1 - \epsilon}} \cdot \theta + \text{higher order terms.}$$

Pitowski (1982; Gudder, 1984) showed the following lemma, which we reproduce without proof:

Lemma. Let events be represented by points on the Poincaré sphere. If the probability between two points A and B is a symmetrical function of the angle between them taken to the origin of the sphere [this is the case here], and the inequalities related to the existence of a Kolmogorovian model are fulfilled, the probability function obeys the following inequality:

$$P(\pi/N) \leq 1 - 1/N, \qquad N = 1, 2, \ldots$$

Around the origin, this imposes that, for an analytical function $P(\theta)$, the first derivative in $\theta = 0$ is smaller than or equal to $-1/\pi$.

Now, we showed that this first derivative tends to 0 in the vicinity of the classical limit, proving the theorem.

In the classical case, it can be shown (Durt, 1996a) that the inequalities related to the existence of a Kolmogorovian model are fulfilled and even saturated for triplets of coplanar points on the sphere. The fact that a Kolmogorovian model exists in the classical case was already shown in Section 5.1 by direct use of the Bayes axiom.

6. THE η-ϵ-MODEL AND THE ACCARDI–FEDULLO INEQUALITIES

6.1 The η-ϵ-Model

It appeared useful to generalize the model when we characterized the lattice of properties associated with it, among other reasons to make the deterministic limit coincident with a Boolean lattice of properties (Aerts and Durt, 1994a; Durt, 1996a). The η-ϵ-model differs from the ϵ-model already defined in the second section in one point: in the ϵ-model, we assumed that the hidden variable ϕ was homogeneously spread over the real interval $[(1 - \epsilon)/2, (1 + \epsilon)/2]$, while in the η-ϵ-model, we assume that it fluctuates inside the interval $[(1-\eta-\epsilon)/2, (1-\eta+\epsilon)/2]$ (with $0 \leq \epsilon \leq 1$ and $\epsilon - 1 \leq \eta \leq 1 - \epsilon$). The interval of fluctuation is now asymmetric around 1/2, the departure from the symmetry being measured by the absolute value of the parameter η. The ϵ-model is in fact a special case of the η - ϵ-model for the symmetrical distribution ($\eta = 0$). We introduce two angles $\theta_{cl,\text{up}}$ and $\theta_{cl,\text{down}}$, which measure the angular opening of the spherical sectors around A_+ and A_- in which the probabilities of getting the answer spin up (with the generalized magnet pointing along the direction A) are respectively 1 and 0. We can now define the η-ϵ distribution. The probability of getting spin up $P(A_+|B_+)$ when the generalized magnet points along the direction A and the state along B depends on the angle θ between A and B as follows:

- $P(A_+|B_+)$ is equal to 1 when $0 \leq \theta \leq \theta_{cl,\text{up}}$, and equal to 0 when $\pi \geq \theta \geq \pi - \theta_{cl,\text{down}}$.
- In between it is a superposition of the two possible results, in a zone of angular opening θ_{sup}. We have then

$$P(A_+|B_+) = \frac{\cos\theta + \cos\theta_{cl,\text{down}}}{\cos\theta_{cl,\text{up}} + \cos\theta_{cl,\text{down}}}$$

The three angles θ_{sup}, $\theta_{cl,\text{up}}$, $\theta_{cl,\text{down}}$ satisfy the following relations: $\cos^2(\theta_{cl,\text{up}}/2) = (1-\eta+\epsilon)/2$, $\sin^2(\theta_{cl,\text{down}}/2) = (1-\eta-\epsilon)/2$, and $\theta_{\text{sup}} + \theta_{cl,\text{up}} + \theta_{cl,\text{down}} = \pi$.

The asymmetry related to the new parameter η implies also that the directions up and down are not considered equivalently. In the η-ϵ-model, the probability of getting the answer spin down with the generalized magnet pointing along the direction A is no longer equal to the probability of getting the answer spin up with the generalized magnet pointing along the direction $-A$ except when $\eta = 0$, which corresponds to the ϵ-model. Effectively, it can be shown that $P(A_+|B_+) = 1 - P(A_-|B_+)$ for all directions A, B on the sphere if and only if $\eta = 0$.

6.2. The η-ϵ-Model and the Accardi–Fedullo Inequalities

Concerning the transition probability, it is easy to generalize the theorem established in section 4, and we have the following theorem (Durt, 1996):

Theorem 6. The η-ϵ-probability does not admit a Hilbert space model unless $\epsilon = 1$ (the quantum case). It never admits a Kolmogorovian model.

With regard to the conditional probability, it is possible to compute it explicitly (Lévêque, 1995) in generalizing the proof sketched in Section 5. It would be too tedious to reproduce here the explicit form of the conditional probability deduced from the η-ϵ-probability so we only reproduce the final result of the classification established in Lévêque (1995) of the η-ϵ conditional probability in terms of the Accardi–Fedullo inequalities. They generalize the results obtained for the ϵ-probability.

(i) The quantum probability is "isolated": it admits a Hilbert space model, but any η-ϵ conditional probability associated to values of ($\theta_{cl,\mathrm{up}}$, $\theta_{cl,\mathrm{down}}$) in the vicinity of (0, 0) (the quantum case) admits neither a Hilbert space model nor (*a fortiori*) a Kolmogorovian model.

(ii) The classical probability ($\theta_{\mathrm{sup}} = 0$) is "isolated": it admits a Kolmogorovian model (when $\theta_{cl,\mathrm{up}} \geq \theta_{cl,\mathrm{down}}$), but any conditional probability associated to values of $\theta_{cl,\mathrm{up}}$, $\theta_{cl,\mathrm{down}}$ in the vicinity of the classical zone does not admit a Kolmogorovian space model. Remark that this is not true for the Hilbert space model: there exists a broad zone surrounding the classical zone for which the conditional probability admits a Hilbert space model, as one can see in Fig. 1.

(iii) When $\theta_{cl,\mathrm{up}} < \theta_{cl,\mathrm{down}}$, the η-ϵ conditional probability admits neither a Hilbert space model nor (*a fortiori*) a Kolmogorovian model.

This last observation is a new feature of the η-ϵ-model and finds its origin in the asymmetry between the directions up and down that we discussed in the remark of the previous subsection. The conditional probability can be interpreted in terms of mutually exclusive and complementary dichotomic experiments only when the distribution of the hidden variable is symmetrical ($\eta = 0$). It can be shown that $P_{cond}(A_+|B_+) = 1 - P_{cond}(A_-|B_+)$ for all directions A, B on the sphere if and only if $\eta = 0$. This corresponds to the special case of the ϵ-model studied in the first part of this work. One can also show (Lévêque, 1995) that $P_{cond}(A_+|B_+) = 1 - P_{cond}(A_-|B_+)$ is an essential condition for the deduction of the inequality Accardi–Fedullo $r \geq -p - q + 1$, as can be verified directly in the proof given in Accardi and Fedullo (1982). This implicit assumption is not only a sufficient condition for the deduction of the inequalities, but it is also (to some extent) a necessary condition, as the two following theorems show (Durt, 1996a):

Theorem 7. If the conditional probability $P_{cond}(A|B) = P_{cond}(\theta)$ vanishes when the angle θ between A and B is π, and if it admits a Kolmogorovian model or a Hilbert space model, then it fulfills the following relation:

$$\forall\theta \in [0, \pi]: \quad P_{cond}(\theta) = 1 - P_{cond}(\pi - \theta) \tag{6}$$

Proof. Let us choose A, B, C coplanar such that the angles between A and B, between A and C, and between C and B are respectively equal to π, θ, and $\pi - \theta$. Then $p = P_{cond}(A|B) = 0$, $q = P_{cond}(B|C) = P_{cond}(\pi - \theta)$, $r = P_{cond}(A|C) = P_{cond}(\theta)$. The Accardi–Fedullo inequalities imply that $|q - 1| \le r \le 1 - |q|$ in the Kolmogorovian case, and that $(-\sqrt{1 - q})^2 \le r \le (+\sqrt{1 - q})^2$ in the Hilbert case. In any case, $r = 1 - q$, proving the theorem. The conditional probability $P_{cond}(A|B)$ between two antipodal points A and B is zero whenever $\theta_{cl,\text{up}} \le \theta_{cl,\text{down}}$ because then the transition probability is zero in the sector of opening $\theta_{cl,\text{down}}$ around the antipodal point, and thus is also zero in the sector of opening $\theta_{cl,\text{up}}$ around it, on which we integrate to get the conditional probability. Whenever $\theta_{cl,\text{up}} \le \theta_{cl,\text{down}}$, the previous theorem thus imposes severe restrictions on the possibility of the existence of a Kolmogorovian model or a Hilbert space model representing the conditional probability. They are expressed in the following theorem (Durt, 1996a), which we reproduce without proof.

Theorem 8. If $\theta_{cl,\text{up}} < \theta_{cl,\text{down}}$, the conditional probability admits neither a Kolmogorovian model nor a Hilbert space model.

This corresponds exactly to result (iii) formulated at the beginning of this subsection, and is in agreement with numerical computations. Remark that when $\theta_{cl,\text{up}} = \theta_{cl,\text{down}}$ ($\eta = 0$), the conditional probability obeys

$$\forall\theta \in [0, \pi]: \quad P_{cond}(\theta) = 1 - P_{cond}(\pi - \theta) \tag{7}$$

and the conditional probability can be interpreted in terms of mutually exclusive and complementary dichotomic experiments. If we drop in the formulation of Accardi–Fedullo's theorem the constraint that A_+ and A_- are dichotomic and mutually exclusive events $[P_{cond}(A_+|B_+) = 1 - P_{cond}(A_-|B_-)]$, we can reproduce the proof of Accardi–Fedullo partially, deducing three inequalities instead of four (Lévêque, 1995). This corresponds to the existence of what we in the next section will call a generalized Kolmogorovian model (Lévêque, 1995; Durt, 1996a).

6.3. The Generalized Kolmogorovian Model

Definition. Let us consider three experiments, not necessarily dichotomic, A, B, C, with the outcomes A_+, A_-, B_+, B_-, C_+, C_-, and the conditional probabilities between them. The conditional probabilities are said to admit a generalized Kolmogorovian model iff

- There exists a probability space that is characterized by a triple (Ω, Σ, μ) where Ω is a nonempty set, Σ the σ-algebra of subsets of Ω, and μ a probability measure on Σ.
- To each experiment we can associate two measurable subsets of Ω (for instance, for A we have A_+, A_-, not necessarily complementary or disjoint).
- The conditional probability is given by the Bayes formula: $P(A_+|B_+) = \mu(A_+ \cap B_+)/\mu(B_+)$.

The generalization of Accardi–Fedullo's theorem is given by the following theorem (Lévêque, 1995):

Theorem 9. If the conditional probability is symmetrical in its arguments, it admits a generalized Kolmogorovian model iff the three conditional probabilities p, q, r fulfill the inequalities

$$p + q - r \leq 1$$

$$p - q + r \leq 1$$

$$-p + q + r \leq 1$$

The numerical results show (Lévêque, 1995) that these three inequalities are fulfilled only in the classical (deterministic) case ($\theta_{sup} = \epsilon = 0$) for all values of η. This is confirmed by the following theorem (Lévêque, 1995):

Theorem 10. The conditional probability violates the three inequalities related to the existence of a generalized Kolmogorovian model for all values of ϵ inside an open interval lower bounded by zero (the classical case); when $\epsilon = 0$, the inequalities are saturated.

This theorem generalizes the corresponding theorem of the previous section according to which, in the ϵ-model ($\eta = 0$), the four inequalities are saturated when $\epsilon = \theta_{sup} = 0$ and for ϵ inside a small open interval lower bounded by 0, the conditional probability violates the inequalities related to the existence of a Kolmogorovian model. From the last theorem, we can deduce that a generalized Kolmogorovian model exists for the classical conditional probabilities ($\epsilon = 0$). In fact we shall do more; we shall explicitly build such a model.

6.4. The Generalized Kolmogorovian Model of the Classical Conditional Probability.

For all the classical Aerts-conditional probabilities of the ϵ-η-model, we can build a generalized Kolmogorovian model, according to the definition given at the beginning of this section, as follows (Durt, 1996a): Let us take the sphere as the state space Ω, and as measure μ the normalized surface on the sphere. To each experimental result represented by a point A of the sphere we can associate a measurable subset of Ω, in fact, the spherical sector of opening $\theta_{cl,\text{up}}$ around A such that the conditional probability is given by the Bayes formula: for instance, $P_{cond}(A|B) = \mu(A \cap B)/\mu(B) =$ (surface of the intersection of a sector of opening $\theta_{cl,\text{up}}$ around A with a sector of opening $\theta_{cl,\text{up}}$ around $B)/2\pi[1 - \cos(\theta_{cl,\text{up}})]$. It is easy to check that, in the classical case, when ϵ equals zero ($\theta_{cl,\text{up}} = \pi - \theta_{cl,\text{down}}$), this is the expression of the conditional probability. This generalized Kolmogorovian model appears to be Kolmogorovian in the sense of Accardi and Fedullo only in the symmetrical situation ($\eta = 0$, the ϵ-model) that we studied in detail in Sections 5.1 and 5.2. The Kolmogorovian model presented there is a special case of the generalized Kolmogorovian model presented here, when we impose that $\eta = 0$.

7. CONCLUSION

The discrepancy between the classical and quantum formalism is obvious in several structural approaches (lattice theory, algebraic approach, convex approach, and so on) that were constructed in order to unify (or to clarify the relation between) these two theories. Although a simple and naive model like the one we presented here cannot solve the problem of reconciling quantum and classical approaches, we do believe that it may serve the purpose of clarifying some of the issues in the debate concerning quantum and classical probability. A first, perhaps somewhat obvious remark pertains to the difference between a classical mechanical theory and a classical statistical theory. In the former, probabilities (if one really wants to introduce them) are either 0 or 1. In the latter, probabilities must be representable as a normalized measure on a σ-algebra of events. In the ϵ-model, we find a direct counterpart of these concepts: the former corresponds to the transition probability, the latter to the conditional probability, both taken in the fluctuationless case. A perhaps more important issue at stake is the question of interpretation of the violation of the inequalities. This question is also related to Bell's inequalities (Bell, 1964), which can be shown to be a set of classically derived inequalities (Gutkowski-Masotto, 1974; Pitowski, 1989). In the literature one can find many different attitudes that are being adopted toward the interpretation of the experimental violation of statistical inequalities. The fact that the probabilities which appear in quantum experiments (for instance, in Stern–Gerlach experiments) admit a Hilbert space model but no Kolmogorovian model in the sense of Accardi and Fedullo is considered by some as an experimental proof that the probability appearing in quantum mechanics is not explainable in terms of simple, understandable models. For others it means that the axioms of Kolmogorov are not fulfilled in nature. In order to better understand the relevance of these attitudes, let us develop, on a metaphorical level, an analogy with the situation in geometry. Imagine three "flat" beings living on the surface of a large sphere. Let us assume that they are convinced of the fact that plane Euclidean geometry is the relevant geometrical structure of their environment. For instance, all the triangles that they ever (locally) studied fulfill the condition that the sum of their three angles is equal to π. Suppose that these three beings do not live close to each other, but have visited each other quite frequently, thereby establishing the shortest path to each other. One day they may decide to measure the angles between the roads they have followed. It is only when all three of them compare their experimental results that they will find themselves in a paradoxical situation (the sum of the angles differs from π). First they might think that they did not establish the shortest route or did not measure the angles precisely enough, but by increasing the accuracy of their measurements they will become more and more convinced something deeper is going on. Eventually they may come to reject the flat Euclidean structure of their environment and question the parallel postulate. Later, they might find that their flat world is the surface of a sphere in a three-dimensional Euclidean space in which the parallel postulate holds once more. What is now the analogy with the probabilistic structure that we encounter in our model?

Of course, if we calculate the probability related to a single outcome, this isolated calculation related to only one experiment follows the Kolmogorovian scheme. For instance, the probability that our hidden variable resides in an interval can be expressed as a measure on a Borel set. The normalization of this measure (Section 2.1) is due to an implicit use of the Bayes axiom. The reasoning used to describe the probabilistic behavior of the elastic is classical and in accordance with Kolmogorovian probability. For instance, it could be approached with arbitrary accuracy by the probability generated by throwing a weighted coin. More generally, it can be proved that if we consider only one observable, in which case all projectors related to the outcome channels commute, the probability is Kolmogorovian (Ballentine, 1986). Nevertheless, we are not in contradiction with the conclusions of Accardi and Fedullo, which primarily classify the collection of probabilities related to different experiments and hence do not treat the case of a single experiment. If we pursue the geometrical comparison, the probabilities related to a single outcome correspond to the local geometry on the sphere which is Euclidean.

If we consider, however, the structure of several different experiments performed on one entity, each of them with an eventual local lack of knowledge, then all these probabilities together form, in general, a non-Kolmogorovian structure. This is indeed the situation in Hilbert space, where the probability measure connected to one experiment is Kolmogorovian, but the whole probability structure, including the transition and conditional probability, connecting different (noncompatible) measurements, is non-Kolmogorovian.

Let us pursue the geometrical analogy and seek if there is a counterpart in axiomatic probability of the possibility of embedding of the non-Euclidean geometry of the sphere inside a three-dimensional geometry. It can be shown (Bana and Durt, 1997) that if we consider the frequencies obtained by averaging quantum frequencies of different experiments with the frequency of realization of these experiments, we also obtain a Kolmogorovian representation for the probabilities connected to these frequencies. This result provides the sought counterpart.

The geometrical analogy suggests, if we consider the very fruitful theories which were obtained thanks to generalizations of the parallel postulate, that one might try to replace one or more of the axioms of the Kolmogorovian scheme. The "easiest way out," according to Suppes (1966), is to replace the σ-algebra of subsets by the weaker condition of σ-additivity. This line of research was pursued further by Gudder (1969, 1979, 1984, 1988) in what he calls quantum probability spaces. Other investigators who tried to enlarge probability theory include Pitowski (1982), concentrating on nonmeasurable sets, and Accardi (1984), focusing on the Bayes axiom. However, up to the present we know of no set of axioms with a clear physical interpretation which solves the problem, the reason being that these extensions are derived mainly from mathematical principles. Perhaps the closest to what we have in mind is the operational statistics of Randall and Foulis (1972, 1973, 1976).

Let us now come back to the hidden variable approach. An essential, nonclassical, feature of our model is that the measuring apparatus is characterized by uncontrollable fluctuations, which leads to a situation of lack of knowledge of the quantum system. Note that our model gives a very simple counterexample of the commonly accepted opinion according to which probabilities due to a lack of knowledge are necessarily Kolmogorovian. An essential difference between our model and statistical mechanics is that we emphasize the irreducible role played by the measuring apparatus. To sidestep the assumptions made by Accardi and Fedullo, other options exist. Durt (1998) has emphasized the role of the preparation as sufficient to evade Accardi and Fedullo's assumption. Czachor (1992) has especially stressed the importance of the state transition as a result of the measurement. The reason for incorporating these features is that in quantum mechanics the observation itself is not a passive act, but rather an active process, literally, an interaction. This is not the case in Kolmogorov's theory of probability, where an observation can be thought of as merely a filter on an initial distribution. In this sense, the violation of classical inequalities shows the end of an old classical paradigm: the myth of the external observer.

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